The orbit space of the action of gauge transformation group on connections

MARIA C. ABBATI, RENZO CIRELLI AND ALESSANDRO MANIÀ
Dipartimento di Fisica, Università di Milano, Italy
Istituto di Fisica Nucleare, Sez. di Milano

Abstract. *The behaviour of orbits of the action of the group of smooth gauge transformations on connections for a principal bundle $P(M,G)$ is discussed with and without compactness assumption on M and G. In the case of compact M and with suitable conditions on G a stratification structure for the space of orbits is established. A natural tame weak Riemannian metric is given on each stratum.*

I. INTRODUCTION

Suitable smoothness structures have been introduced on the gauge transformation group $\mathcal{G}$ and on the space $\mathcal{C}$ of connections of a principal bundle $P(M,G)$. Properties of $\mathcal{G}$ and $\mathcal{C}$ can be investigated with and without assumptions on M and/or G ([1], [2] and references therein).

A natural development is studying the orbit space of this action which is also a relevant object from the physical point of view in gauge theories. It is well known that the singular Yang-Mills Lagrangians can be treated as non singular Lagrangians if one introduces the true configuration space, i.e. the quotient $\mathcal{C}/\mathcal{G}$ of the space of connections under the action of the gauge transformation group.

The common attitude has been to introduce some technical assumptions in order to obtain that the orbit space is a (infinite dimensional) manifold. Thus classical Yang-Mills theories can be regarded as (infinite dimensional) dinamical systems on $\mathcal{C}/\mathcal{G}$ governed by a Lagrangian effectively defined on it [3]. The kinetic energy term of the Lagrangian

Key-Words: Gauge theories, orbit space, connections.
1980 MSC: 53 C 05, 53 C 80, 57 N 89, 81 E 13.

is provided by a Riemannian metric naturally defined on the orbit space.

On the other hand, topological and geometric aspects of the orbit space and of the group of gauge transformations appeared very early of relevance in quantization problems, mainly in the path integral approach. At first Singer [4] proved the impossibility of a global gauge fixing. Local gauge fixing is however assured and is interpreted as the existence of local sections of the projection $\mathcal{C} \to \mathcal{C}/\mathcal{G}$. More recently, it has been pointed out that many physical properties of gauge fields are related to non trivial topological or geometrical aspects. A natural metric on the orbit space would allow a geometric interpretation of Faddev-Popov determinant [5], [6] and it is expected to be an useful tool in a continuous non-perturbative regularization of Yang-Mills path integral [7].

As a matter of fact, the geometry of orbit space is intricated; many assumptions are commonly made to obtain on it the structure of Hilbert manifold. The spaces of mappings involved in the theory are assumed to be defined on a compact manifold (the compactification of space-time or of ordinary space) and to belong to some convenient Sobolev completion. The heavy – but technically very useful – compactness condition can be justified by assumptions on the behaviour of fields at infinity which appear very reasonable from the physical point of view. The physical meaning of the Sobolev completion is hardly testable and in fact this procedure can be avoided. The further usual assumption that the gauge group is compact covers all the usual Yang-Mills theories.

But the most relevant point is that the orbit space fails in general to be a manifold [4]: even under the above compactness assumptions one can only define on it a stratification structure where a (weak) Riemannian structure is defined only on each stratum. What is commonly considered is just the generic stratum which is open and dense in the entire orbit space. In our opinion, however, the complete stratified structure of orbit space should be taken into account in the analysis of problems related with gauge fixing and path integral quantization. We emphasize that this intricated situation is not peculiar of gauge theories. For instance, it is well known that stratification structures appear in general relativity [8], [9], [10], [11].

In this paper our aim is to show how the structure of stratification on the orbit space can be worked out in a context as general as possible. The first problem one meets is to guarantee nice topological properties of stability subgroups and orbits. Yet, investigation of this topological aspect shows that the case of non compact manifold M is hardly workable and therefore we are forced to retire to the compact case.

With compactness assumption we are also gratified by the structure of tame Fréchet Lie group and tame Fréchet manifold on $\mathcal{G}$ and $\mathcal{C}$, respectively. The important point is that in this context one can dispose of a workable version of the inverse function theorem [12]. This allows us to show that orbits are nicely immersed as submanifolds in $\mathcal{C}$, under the further assumption that the group G essentially is compact.

On the basis of these topological properties and tameness properties of the manifolds under consideration, we can run the classical route to construct a stratification structure which goes through a slice theorem. This is the procedure followed in [8], [9], [10], [11], [13], [14] and [15] jointly with Sobolev space techniques to handle infinite dimensional manifolds acted by infinite dimensional groups. Other methods have been used in the study of the orbits of the solutions of Yang-Mills equations [16], [17].

In Sec. II we investigate stability subgroups and we show that they are locally compact Lie subgroups of $\mathcal{G}$ in the case of compact M and discrete subgroups in the non-compact case.

In Sec. III we show the tameness properties of $\mathcal{G}$ and $\mathcal{C}$ in the case of compact M. This is linked with the problem of disposing of an inverse function theorem. In Sec. IV we investigate the behaviour of orbits. Further assumptions are made to garantee that orbits are closed. In this case orbits are embedded submanifolds in $\mathcal{C}$. In Sec. V a slice theorem is established and in Sec. VI the wanted stratification of the orbit space is obtained. We are also able to introduce a natural tame weak Riemannian metric on each stratum.

Throughout the paper we assume M to be an ordinary connected manifold and G an ordinary connected Lie group.

II. STABILITY SUBGROUPS

Throughout the paper $P(M,G) = (P,p,M;G)$ will denote a principal bundle, for which M is an ordinary connected manifold and G an ordinary Lie group, with Lie algebra $\mathfrak{g}$. The group $\mathcal{G}$ of gauge transformations is the set $\operatorname{Sec} P[G]$ of smooth sections of the associated bundle $P[G] = (P \times_G G, p_G, M)$, the action of G onto itself being defined by inner automorphisms. It has been proved in [1] that $\mathcal{G}$, endowed with the FD-topology, is a NLF-Lie group and that its Lie algebra $L(\mathcal{G})$ is the NLF-space (inductive limit of nuclear Fréchet spaces) $\operatorname{Sec}_c P[\mathfrak{g}]$ of compact support sections of the adjoint bundle $P[\mathfrak{g}] = (P \times_G \mathfrak{g}, p_G, M)$. The exponential map of $G, \exp : \mathfrak{g} \to G$ induces a fiber preserving map $\widetilde{\exp} : P[\mathfrak{g}] \to P[G]$. The map $(\widetilde{\exp})_* : \operatorname{Sec}_c P[\mathfrak{g}] \to \operatorname{Sec} P[G]$ is the exponential map of the gauge transformation group, and we will call it Exp. Note that Exp is a local diffeomorphism [1].

For a fixed $u_0 \in P$, we can define a map $E_0 : \mathcal{G} \to G$ by

$$s(p(u_0)) \doteq [(u_0, E_0(s))]$$

where by $[(u,g)]$ we denote the element of $P \times_G G$ containing (u,g). One easily recognizes that E_0 is a smooth map and a group homomorphism.

We consider the space $\mathcal{C}$ of principal connections as a local manifold, with local model the NLF-space $\mathcal{A} = \operatorname{Sec}_c L(TM, P[\mathfrak{g}])$ and recall that the natural left action

$$A : \mathcal{G} \times \mathcal{C} \to \mathcal{C}, A(s,\gamma) = (s^*)^{-1}\gamma$$

is smooth [2]. Briefly, we denote $A(s,\gamma)$ with $s\gamma$. As starting point in investigating the structure of the orbit space of this action, we study stability subgroups.

Let S_γ be the stability subgroup of the connection γ. It is well known that the elements of S_γ are the γ-parallel sections of $P[G]$. As M is connected, these sections are determined by their value at a point, say x_0, of M. Indeed, let $c : \mathbb{R} \to M$ be a smooth curve joining x_0 to x, u_0 a point over x_0 and $Pt(c,t,u_0)$ the parallel transport along the curve c starting from u_0. We have for $s \in S_\gamma$

$$s(x) = [\,(Pt(c,1,u_0), E_0(s))\,]. \tag{2.1}$$

Therefore the map E_0, if restricted to S_γ, is an injective homomorphism and, as is well known, its image is $C(\mathcal{H}_{u_0})$, the centralizer in G of the holonomy group $\mathcal{H}_{u_0}$ of the connection γ.

If M is not compact and $s \in S_\gamma \cap \mathcal{G}_c$, where $\mathcal{G}_c = \mathrm{Sec}_c\, P[G]$, there exists $u_0 \in P$ such that $E_0(s) = E_0(e)$, where e is the unit of $\mathcal{G}$, so that $E_0(s)$ equals the unit e_G of G. Thus, formula (2.1) implies $S_\gamma \cap \mathcal{G}_c = \{e\}$ (so that the group $\mathcal{G}_c$ acts freely on $\mathcal{C}$). Moreover, we recall that $\mathcal{G}_c$ is an open-closed normal subgroup of $\mathcal{G}$. Thus S_γ is a discrete subgroup of $\mathcal{G}$, acting discontinuosly on $\mathcal{G}$ and the quotient space $\mathcal{G}/S_\gamma$ admits a structure of quotient manifold modelled on the same model of $\mathcal{G}$, i.e. on the Lie algebra $L(\mathcal{G}) = \mathrm{Sec}_c\, P[\boldsymbol{g}]$. Things are very different if M is compact. In this case the FD-topology on $\mathcal{G}$ and S_γ agrees with the Whitney topology and it is rather standard to prove that the stability subgroup of a connection γ is isomorphic as topological group to $C(\mathcal{H}_{u_0})$. Would $\mathcal{G}$ be an ordinary Lie group one could immediately infer from this that S_γ is a Lie group, Lie isomorphic to $C(\mathcal{H}_{u_0})$. Actually this is true also in the present case, but it needs a non trivial proof, even if the group S_γ is a locally compact subgroup of $\mathcal{G}$, since for Fréchet Lie groups one cannot at present dispose of a general theorem concerning Lie subgroups.

To this purpose, we introduce

$$K = \{\chi \in L(\mathcal{G}) \mid \mathrm{Exp}(t\chi) \in S_\gamma, \forall t \in \mathbb{R}\}.$$

Since Baker-Campbell-Hausdorff formula holds in $\mathcal{G}$ and S_γ is closed we see that K is a Lie subalgebra of $L(\mathcal{G})$. It is not difficult to prove that the homomorphism $E_0' : L(\mathcal{G}) \to \boldsymbol{g}$ induced by E_0, if restricted to K gives a isomorphism of K with the Lie algebra $\boldsymbol{g}_0$ of $C(\mathcal{H}_{u_0})$. Moreover one easily sees that $L(\mathcal{G}) = K \oplus H$, where H denotes the inverse image by E_0' of any complement $\boldsymbol{h}$ of $\boldsymbol{g}_0$ in $\boldsymbol{g}$.

Let $\phi : K \oplus H \to \mathcal{G}$ defined by $\phi(\chi,\lambda) = \mathrm{Exp}\,\chi\,\mathrm{Exp}\,\lambda$.

In the proof that S_γ is a Lie subgroup of $\mathcal{G}$, a crucial step is to prove that there exists an open neighborhood $\mathcal{V}_0$ of the zero and an open neighborhood $\mathcal{U}_e$ of the unit of $\mathcal{G}$ such that $\phi : \mathcal{V}_0 \to \mathcal{U}_e$ is a diffeomorphism. In the theory of ordinary Lie groups the analogous statement is proved as a consequence of the inverse function theorem. Therefore the map $\varphi : \mathbf{g}_0 \oplus \boldsymbol{h} \to G, \varphi(k,h) = \exp k \exp h$ is a diffeomorphism from an open neighborhood of the zero and an open neighborhood U_{e_G} of the unit e_G of G. One can ever assume that

i) $U_{e_G} U_{e_G} \subset U_{e_G}$,

ii) $U_{e_G} U_{e_G}^{-1} \subset U_{e_G}$ and that

iii) on U_{e_G} the inverse of the map exp is defined.

Then define $P[U_{e_G}] = \{[(u,g)] \in P[G], g \in U_{e_G}\}$ and $\mathcal{U}_e = \{s \in \mathcal{G} \mid s(x) \in P[U_{e_G}], \forall x \in M\}$. By [18], Sec 4, $\mathcal{U}_e$ is an open subset of $\mathcal{G}$. Of course, $\mathcal{U}_e$ satisfies the analogues of i), ii) and iii).

For $s \in \mathcal{U}_e, E_0(s) = \varphi(k,h)$ for a unique pair $(k,h) \in \mathbf{g}_0 \oplus \boldsymbol{h}$ and a unique χ exists in K such that $E_0'(\chi) = k$. Put $s_0 = \operatorname{Exp}\chi$ and $s_1 = s_0^{-1}s$. Then s_0 and s_1 belong to $\mathcal{U}_e$. Since $s_1 = \operatorname{Exp}\lambda$ implies $\lambda \in H$, we see that $s = \phi(\chi,\lambda)$, with $\chi \in K$ and $\lambda \in H$. The map $s \mapsto (\chi,\lambda)$ is the inverse on $\mathcal{U}_e$ of the map ϕ and is smooth, since it is the composition of smooth maps. Let $\mathcal{V}_0 = \phi^{-1}(\mathcal{U}_e)$. Now, to prove that S_γ is a splitting submanifold of $\mathcal{G}$ it is enough to show that $\exp(\mathcal{V}_0 \cap K)$ is neighborhood of the identity in S_γ. But this follows by the above arguments and by adapting the classical arguments used in the proof of Cartan Theorem [19].

We have obtained therefore the expected result.

THEOREM 2.1. *Let M be a compact connected manifold. For every connection γ, the stability subgroup is a finite dimensional splitting Lie subgroup of $\mathcal{G}$, Lie-isomorphic to the centralizer of the holonomy group of γ.* ∎

III. TAMENESS PROPERTIES OF GAUGE TRANSFORMATION GROUPS

It is well known that dealing with manifolds modelled on locally convex vector spaces more general that Banach spaces one is faced with the difficulties arising from the lack of inverse map theorem. Perhaps for this reason it is a common use in physical applications to retire to Banach manifolds. However, a workable version of the inverse map theorem (Nash-Moser Theorem) is now availabe for a significant subcategory of Fréchet space called «tame Fréchet spaces» by Hamilton [12]. If tame Fréchet manifolds are accordingly defined, the Nash-Moser Theorem can be formulated as follows (see III.1.1.1. of [12]).

NASH-MOSER THEOREM. *Let X and Y be tame Fréchet manifolds and $f : U \subset X \to Y$ a smooth tame map on the open subset U of X. Suppose that $T_x f : T_x X \to T_{f(x)} Y$ is bijective for every $x \in U$ and that*

$$Vf : f^*TY \to TX, (x, v_{f(x)}) \mapsto (T_x f)^{-1} v_{f(x)}$$

is smooth tame. Then f is locally invertible and each local inverse is smooth tame (i.e. f is a tame local diffeomorphism at each point of U). ∎

A simple corollary of the theorem which we will use in the next section is the following.

COROLLARY. *Let the map $f : U \subset X \to Y$ be smooth tame and satisfy the following conditions:*

1) $T_x f : T_x X \to T_{f(x)} Y$ *is injective for every $x \in V$;*
2) $\mathrm{T} = \{\mathrm{Im}\, T_x f\}(x \in U)$ *is a tame subbundle of $f^* TY$,*
3) *the bundle map of the left inverses $V_f : \mathrm{T} \to TX_{\lceil U}$ is smooth tame.*

Then f is a tame immersion at every point of U. ∎

We recall that f is a **tame immersion** at x if there exists an open neighborhood U of x such that $f_{\lceil U}$ is a tame diffeomorphism with a tame splitting submanifold of Y.

Now, if the base manifold M is assumed to be compact, the group $\mathcal{G}$ of gauge transformations clearly becomes a Fréchet-Lie group. Actually, we can show that it is a tame Fréchet-Lie group and that its exponential map is smooth tame. In fact, by Theorem II.2.3.1 of [12], $\mathcal{G}$ is a tame manifold and to prove tameness of group operations and of the exponential map is simply a matter of decomposing them as in [1] and the using Theorem II.2.3.3 of [12].

The tameness properties of $\mathcal{G}$, the properties of S_γ stated in theorem 2.1 and local properties at zero of the map $\phi : K \oplus H \to \mathcal{G}$ above defined allows us to prove that, in the case of compact M, there exists on $\mathcal{G}/S_\gamma$ a unique smooth structure such that $(\mathcal{G}, \pi_\gamma, \mathcal{G}/S_\gamma; S_\gamma)$ is a smooth tame principal bundle (π_γ is the canonical projection).

Similarly, we can prove that the space C of connections is a splitting affine subspace of a tame Fréchet space and the action A of $\mathcal{G}$ on C is smooth tame.

The notion of tame Fréchet-Lie group could be relevant in the analysis of the structure of $\mathcal{G}$ even in the case of non compact M as it is shown by the following arguments.

Let $K_0 \subset K_1 \subset K_2 \subset \ldots$ be an increasing exhaustion of M by compact sets K_n with $K_n \subset \mathrm{int}\, K_{n+1}$. Choose $f_n \in C^\infty(M)$,

$$0 \le f_n(x) \le 1,\ f_n(x) = 1 \text{ on } K_n \text{ and } f_n(x) = 0 \text{ on } M \setminus K_{n+1}$$

By Sard's Theorem the regular values of f_n are dense, so there is a regular value $y_n \in (0,1)$. Put $L_n = f_n^{-1}([y_n, 1])$; then L_n is a submanifold with boundary of M, with $\partial L_n = f_n^{-1}(y_n)$ and $K_n \subset \mathrm{int}\, L_n \subset L_n \subset \mathrm{int}\, K_{n+1}$ for all n. So we may assume that the original exhaustion $K_n \subset K_{n+1} \subset \ldots$ consists of submanifolds with boundary of M of dimension equal to the dimension of M.

The space $\mathcal{D}(M, K_n)$ of the smooth functions on M with support contained in K_n is a tame Fréchet space by II, Corollary 1.3.8 of [12]. Also the embedding $\mathcal{D}(M, K_n) \to \mathcal{D}(M, K_{n+1})$ is tame, but $\mathcal{D}(M, K_n)$ is not a direct summand in $\mathcal{D}(M, K_{n+1})$.

Similarly, $\mathcal{G}_{K_n} = \{$ elements of $\mathcal{G}$ with support in $K_n\}$ are tame Fréchet-Lie groups and $\mathcal{G}_c$ is the inductive limit (in the category of topological groups) of the tame Fréchet-Lie groups $\mathcal{G}_{K_n}$'s.

IV. THE ORBITS OF THE GAUGE ACTION

It is well known that behind the case of compact groups acting on compact manifolds, there can be very anomalous behaviour of the orbits even for finite dimensional Lie groups and manifolds. Fairly good properties of the orbit space require at least local closedness of orbits. For the action $\mathcal{A} : \mathcal{G} \times \mathcal{C} \to \mathcal{C}$ this condition is satisfied in many relevant cases, the most relevant being the case of a compact base manifold and a compact structure group. Under these conditions the orbits are even closed.

THEOREM 4.1. *Let M and G be compact. The orbits of the action A are closed.*

Proof. Let $\mathcal{O}_\gamma \subset \mathcal{C}$ be the orbit through γ and $\{\gamma_n\}$ a sequence in $\mathcal{O}_\gamma$ converging to γ_0. For every integer k the sequence $\{\gamma_n\}$ converges to γ_0 in the Sobolev affine space $\mathcal{C}^{(k-1)}$ of Sobolev $H^{(k-1)}$-connections where the orbits of the action of the group $\mathcal{G}^{(k)}$ of the $H^{(k)}$-sections of $P[G]$ are closed [13]. Therefore there exists $s \in \mathcal{G}^{(k)}$ such that $\gamma_0 = s\gamma$. Remark that γ_0 and γ are smooth sections. This in turn implies that s is smooth, so that $\gamma_0 \in \mathcal{O}_\gamma$. Actually one has, in a local chart, $\omega_0(x) = \mathrm{Ad}_{f(x)}\omega(x) - \mathrm{d}_x f$, where $\mathrm{d}_x f$ is the (right) logaritmic derivative of f and ω_0, ω, f are local representative of γ_0, γ, s respectively. The element $s \in \mathcal{G}^{(k)}$ is C^1 for large k so that f is C^1 and the above relation implies by induction that f is smooth. ∎

The following theorem gives other relevant cases in which the orbits are closed.

THEOREM 4.2. *Let M be compact. In the following cases the orbits of the action A are closed:*

i) *G is the vectorial group $\mathbf{R}^n$;*
ii) *$G = K \times \mathbb{R}^n$, with K a compact Lie group;*
iii) *G is an abelian Lie group.*

Proof. i) For a vectorial group the action is simply given by $\omega(x) - \mathrm{d}_x \hat{f}$. Hence the orbits are homeomorphic to the range of the logaritmic derivative. Since the logaritmic derivative is a differential operator $d : \mathrm{Sec}\, P[\mathbf{R}^n] \to \mathrm{Sec}\, L(TM, P[\mathbf{R}^n])$ with injective symbol and the Fréchet spaces of smooth sections of Riemannian bundles are projective limits of the Sobolev spaces of $H^{(k)}$-sections, from Theorem 3.13 of [20] we easily obtain that the range of d is a closed subspace.

ii) It is well known that, if $G = K \times \mathbf{R}^n$, then $P(M, G)$ is the fibered product of two principal bundles $P(M, K)$ and $P(M, \mathbf{R}^n)$. Accordingly, we have splittings of the gauge transformation group, of the connection space and of the action. By Theorem 4.1. and point i) we obtain that the orbits are closed.

iii) follows by ii) since every abelian (connected) Lie group is a direct product of tori and vectorial groups. ∎

In general one cannot expect that the gauge transformation orbits are closed. In Appendix 2 we give a simple example of locally closed but not closed orbits. Moreover there are examples of gauge transformation orbits which are closed whilst the group G does not satisfy the hypothesis of the above theorem.

Let us denote briefly by A^s and A_γ the reduced maps of the action A, defined by fixing $s \in \mathcal{G}$ or $\gamma \in \mathcal{C}$, respectively. The commutative diagram

$$\begin{array}{ccc} \mathcal{G} & \overset{A_\gamma}{\rightarrow} & \mathcal{C} \\ {\scriptstyle \pi_\gamma}\downarrow & {\scriptstyle i_\gamma}\nearrow & \\ \mathcal{G}/S_\gamma & & \end{array}$$

defines a smooth map i_γ.

Clearly, i_γ is bijective onto $\mathcal{O}_\gamma$. In the case of compact M local closedness of orbits is sufficient to garantee that i_γ *is an open mapping onto* $\mathcal{O}_\gamma$. This can be shown by using the open mapping theorem given in Appendix 1. To apply this theorem to our case we remark that $\mathcal{G} = \mathrm{Sec}\, P[G]$ is a Fréchet-Lie group, hence it is a metrizable and complete group; it is separable, since it is a subset of the separable metric space $C^\infty(M, P[\boldsymbol{g}])$; the orbit $\mathcal{O}_\gamma$ is non meagre as a locally closed subspace of the complete metrizable space $\mathcal{C}$. Moreover, from tameness of the action A and the existence of a family of tame local sections of the bundle $(\mathcal{G}, \pi_\gamma, \mathcal{G}/S_\gamma; S_\gamma)$ we obtain easily that i_γ is smooth tame.

We can go further and prove that i_γ is a *tame closed embedding* under the further assumption that the Lie algebra $\boldsymbol{g}$ admits an Ad-invariant scalar product $(\,|\,)_{\boldsymbol{g}}$. This assumption implies that the image of G under the adjoint representation is a compact group, since it is a closed subgroup of the orthogonal group of $\boldsymbol{g}$ and this amounts to say that G satisfies condition ii) of Theorem 4.2. Of course this requirement is rather restrictive since, for instance, semisimple non compact groups and other interesting groups are excluded.

From now on, we assume however that M and G satisfy the above conditions.

Making use of the scalar product $(\,|\,)_{\boldsymbol{g}}$ we first make the vector bundle $P[\boldsymbol{g}]$ into a

Riemannian bundle defining

$$(h_x|k_x)_x := (h|k)_{\mathbf{g}}$$

if $h_x = [(u,h)], k_x = [(u,k)]$ and $p(u) = x$, and then, using also a Riemannian metric on M, we endow $L(\mathcal{G}) = \text{Sec}\, P[\mathbf{g}]$ of a weak scalar product $\langle | \rangle$ defined by

$$\langle \lambda | \eta \rangle = \int_M (\lambda_x | \eta_x)_x \mathrm{d}\, m \tag{4.1}$$

where m is the positive measure defined by the Riemannian metric on M. Analogously, using the Riemannian metric on M, we make $T^*M \otimes P[\mathbf{g}]$ into a Riemannian bundle and endow the tame Fréchet space $\mathcal{A} = \text{Sec}\, L(TM, P[\mathbf{g}]) = \text{Sec}(T^*M \otimes P[\mathbf{g}])$ of a weak scalar produt $(|)$ defined by

$$(\alpha|\beta) = \int_M (\alpha(x)|\beta(x))_x \mathrm{d}\, m \tag{4.2}$$

where m is the positive measure defined by the Riemannian metric on M. Obviously this scalar product on $\mathcal{A}$ can be understood, when needed, as a weak Riemannian metric on the tame affine manifold $\mathcal{C}$.

A natural smooth tame action Ad of $\mathcal{G}$ on $\mathcal{A}$ can be defined. Let Ad : $\mathcal{G} \times L(\mathcal{G}) \to L(\mathcal{G}), (\text{Ad}_s \lambda)(x) := \widetilde{\text{Ad}}_{s(x)} \lambda(x)$ be the adjoint action of $\mathcal{G}$ on $L(\mathcal{G})$, where $\widetilde{\text{Ad}}$ is the bundle map

$$\widetilde{\text{Ad}} : P[G] \times_M P[\mathbf{g}] \to P[\mathbf{g}]$$

$$([(u,g)],[(u,h)]) \to [(u, \mathbf{Ad}_g\, h)].$$

If $\alpha \in \mathcal{A}$ is of the form $\alpha = a \otimes \lambda$, with a scalar 1-form a on M and $\lambda \in L(\mathcal{G})$, $\mathbf{Ad}$: $\mathcal{G} \times \mathcal{A} \to \mathcal{A}$ can be defined setting

$$\mathbf{Ad}_s\, \alpha := a \otimes \mathbf{Ad}_s\, \lambda.$$

Note that this action is tame linear (since it is just the composition with the smooth bundle map $\widetilde{\text{Ad}}$) and orthogonal, namely it leaves the weak scalar product (4.2) invariant.

Obviously enough

$$(T_\gamma A^s)(\gamma, \alpha) = (s\gamma, \mathbf{Ad}_s\, \alpha), s \in \mathcal{G}, \alpha \in \mathcal{A}. \tag{4.3}$$

Using this jointly with the very well known relation between the tangent of A_γ and the covariant derivative ∇_γ defined by γ, namely

$$T_e A_\gamma(e, \lambda) = (\gamma, \nabla_\gamma \lambda), \lambda \in L(\mathcal{G}), \tag{4.4}$$

we obtain

$$\nabla_{s\gamma} \text{Ad}_s \lambda = \mathbf{Ad}_s\, \nabla_\gamma \lambda. \tag{4.5}$$

THEOREM 4.3. *Let the tame Fréchet space $\mathcal{A}$ be equipped with the scalar product (4.2). Then* $\mathrm{Im}\,\nabla_\gamma$ *is an* S_γ *invariant tame orthogonal splitting subspace, i.e.*

$$\mathcal{A} = \mathrm{Im}\,\nabla_\gamma \oplus (\mathrm{Im}\,\nabla_\gamma)^\perp .$$

Proof. We use elliptic theory and the well known fact that ∇_γ is a differential operator with injective symbol. The Kodaira decomposition was proved by [20] in the case of Sobolev spaces of $H^{(k)}$-sections of vector bundles. His proof can be easily implemented to Fréchet spaces of smooth sections, by using the fact that these spaces are tame Fréchet spaces in the grading of $H^{(k)}$ Sobolev norms [12]. To prove that the subspaces of the decomposition are tame subspaces, we remark that they are tame direct summands. The projections on these subspaces are indeed the projective limits of the corresponding orthogonal projections in Sobolev spaces, so that they are tame.

Finally, by (4.5) we get $\mathrm{Im}\,\nabla_{s\gamma} = \mathbf{Ad}_s\,\mathrm{Im}\,\nabla_\gamma$ is S_γ-invariant with respect to the $\mathbf{Ad}$ -action. ∎

We remark that, as in the case of Kodaira decomposition for Sobolev space of $H^{(k)}$ sections of vector bundles, $(\mathrm{Im}\,\nabla_\gamma)^\perp$ is precisely the kernel of $\nabla_\gamma^\dagger$, the formal adjoint of the differential operator ∇_γ with respect to the weak scalar products (4.1) and (4.2).

Let Θ be an open neighborhood of the identity coset o of $\mathcal{G}/S_\gamma$, on which a tame local section χ of the principal bundle $(\mathcal{G}, \pi_\gamma, \mathcal{G}/S_\gamma; S_\gamma)$ is defined and $i_\Theta^* TC$ be the restriction to Θ of the pullback of TC by i_γ.

LEMMA 4.4. *The subset* $\mathrm{T} = \{\mathrm{Im}\,T_\theta i_\gamma\}(\theta \in \Theta)$ *is a splitting tame subbundle of* $i_\Theta^* TC$, *with standard fibre* $\mathrm{Im}\,\nabla_\gamma$.

Proof. As TC is trivial, $i_\Theta^* TC$ is simply $\Theta \times \mathcal{A}$. However, defining $\psi : i_\Theta^* TC \to \Theta \times \mathcal{A}$ by

$$\psi(\theta, \alpha) = (\theta, \mathbf{Ad}_{\chi^{-1}(\theta)}\,\alpha)$$

we get a more convenient trivialization since, as we now show, $\psi(\mathrm{T}) = \Theta \times \mathrm{Im}\,\nabla_\gamma$. If L_s^π is the left translation by $s \in \mathcal{G}$ on $\mathcal{G}/S_\gamma$, by $i_\gamma \circ L_s^\pi = A^s \circ i_\gamma$ we get

$$T_\theta i_\gamma \circ T_o L^\pi_{\chi(\theta)} = T_\gamma A^{\chi(\theta)} \circ T_o i_\gamma . \tag{4.6}$$

Hence using (4.4) and (4.3)

$$\mathrm{Im}\,T_\theta i_\gamma = \{(i_\gamma(\theta), \mathbf{Ad}_{\chi(\theta)}\,\beta\}(\beta \in \mathrm{Im}\,\nabla_\gamma) .$$

Therefore T can be identified with the set of pairs (θ, α) such that $\alpha = \mathbf{Ad}_{\chi(\theta)}\,\beta$ with $\beta \in \mathrm{Im}\,\nabla_\gamma$. ∎

THEOREM 4.5. *The map* i_γ *is a tame immersion.*

Proof. First we prove that i_γ is a tame immersion at the identity coset o. To this aim, we will check that the conditions of Corollary at Nash-Moser Theorem are satisfied in the neighborhood Θ of o, defined before Lemma 4.4. Injectivity of $T_o i_\gamma$ can be proved by an easy computation and injectivity of $T_\theta i_\gamma$ at every point $\theta \in \Theta$ follows by formula 4.6; the second condition is provided by Lemma 4.4.

So, we are left with the proof that the bundle map of the left inverses $V i_\gamma : \mathrm{T} \to T(\mathcal{G}/S_\gamma)_{\restriction\Theta}$ is smooth tame. According to II, Theorem 3.1.1. of [12] it is enough to show that $V i_\gamma$ is continuous and tame. Since

$$(V i_\gamma)(\theta, \alpha) = T_o L^{\pi}_{\chi(\theta)} \circ V_o i_\gamma \circ T_\gamma A^{\chi(\theta)^{-1}}(i_\gamma(\theta), \alpha)$$

where $V_o i_\gamma : \operatorname{Im}\nabla_\gamma \to T_o(G/S_\gamma)$ is the restriction of $V i_\gamma$ to the fiber of T on o, we are reduced to prove that $V_o i_\gamma$ is continuous and tame, and this we can do with the help of projective limit techniques. By the results of [13] we have the following situation: the Fréchet spaces $T_o(\mathcal{G}/S_\gamma)$, $\operatorname{Im}\nabla_\gamma$ and $\mathcal{A}$ are projective limits of sequences of suitable Sobolev spaces $H^{(k)}, \operatorname{Im}^{(k)}_\gamma$ and $\mathcal{A}^{(k)}$ respectively, with $\operatorname{Im}\nabla^{(k)}_\gamma \subset \mathcal{A}^{(k-1)}$, and the map $T_0 i_\gamma$ is the projective limit of a sequence of maps $T_0 i^{(k)}_\gamma : H^{(k)} \to \operatorname{Im}^{(k)}_\gamma$. Their inverse maps $V_o i^{(k)}_\gamma$ are continuous by Open Mapping Theorem so that $\|(V_o i_\gamma)^{(k)}\alpha\|_{k-1} \leq M_k\|\alpha\|_k$.

As $\operatorname{Im}\nabla_\gamma$ is a tame Fréchet space in the grading defined by these Sobolev norms and $V_o i_\gamma = \varprojlim V_o i^{(k)}_\gamma$, we have for $\alpha \in \operatorname{Im}\nabla_\gamma$

$$\|(V_o i_\gamma)\alpha\|_{k-1} = \|(V_o i^{(k)}_\gamma)\alpha\|_{k-1} \leq M_k\|\alpha\|_k$$

so that $V_o i_\gamma$ is a tame continuous map.

Hence i_γ is a tame immersion at o; to show that i_γ is a tame immersion at every point of $\mathcal{G}/S_\gamma$, one can use the transitivity of $\mathcal{G}/S_\gamma$ and the $\mathcal{G}$-invariance of the weak metric. ∎

As a consequence of Theorem 4.2., Theorem 4.3. and Theorem 4.5. we finally obtain the wanted result.

THEOREM 4.6. *Let* M *be a compact connected manifold and* $\mathbf{g}$ *admit an* Ad *-invariant product. Then* i_γ *is a tame closed embedding.* ∎

V. THE SLICE THEOREM

The notion of slice and slice theorems are useful tools in studying quotient spaces with respect group actions. The concept of slice stems from the work of Gleason, Mostow, Palais and others (see [21] and references therein). A brief survey on the subject in the context of differentiable action is given in [11]. Here we adopt their definition of slice.

DEFINITION 5.1. A slice for the action of a Lie group G on a manifold X at a point $x \in X$ with stability group S_x is a submanifold $\mathcal{S}_x$ containing x such that:

i) $\mathcal{S}_x$ is S_x-invariant;

ii) if $g \in G$ and $g \notin S_x$ then $g\mathcal{S}_x \cap \mathcal{S}_x = \emptyset$;

iii) there is a local section $\chi : U \subset G/S_x \to G$ defined in a neighborhood U of the identity coset such that the map

$$\Phi : U \times \mathcal{S}_x \to X, \Phi(u, y) = \chi(u)y$$

is a diffeomorphism onto a neighborhood $\mathcal{V}$ of x.

The existence of a slice at a point $x \in X$ implies that the stability subgroup S_x is locally maximal in the sense that, as consequence of i) ... iii), if $y \in \mathcal{S}_x$ then $S_y \subset S_x$. If $y \in \mathcal{V}$ then S_y is conjugate to a subgroup of S_x, namely there exists a $g \in G$ such that $gS_yg^{-1} \subset S_x$. Thus, if x has a trivial or at least a minimal stability subgroup (remember that in our case S_γ is isomorphic to $C(\mathcal{H}_{u_o}))$, then $S_y = S_x \forall y \in \mathcal{S}_x$ and therefore the slice intersects every orbit through $\mathcal{V}$ only in one point. In fact, as we will see, it can provide a chart at $\mathcal{O}_x$ for the generic stratum of the orbit space.

The construction of a slice at every point x of X goes through the construction of a normal bundle to $\mathcal{O}_x$-usually realized as a bundle orthogonal to the tangent bundle of the orbit made up by means of an invariant metric on X – and through the construction of an equivariant diffeomorphism of this normal bundle with an open neighborhood of the orbit (i.e. one wants an invariant tubular neighborhood of the orbit).

At first we introduce a suitable family of $\mathcal{G}$-invariant metrics on C by means of a $\mathcal{G}$-equivariant family of differential operators D^γ.

Consider a G-invariant metric on P induced by the metric on M and any G-invariant metric on G and denote by H the projection of TP on its subbundle orthogonal to the vertical bundle. For a given connection γ, the differential operator D^γ on $A(M, P[\mathbf{g}])$, the space of the $P[\mathbf{g}]$-valued forms on M, is defined for every $l \in \mathbb{N}$, by

$$D^\gamma : A^l(M, P[\mathbf{g}]) \to A^{l+1}(M, P[\mathbf{g}])$$

$$(D^\gamma\alpha)_x(\xi_x^o, \ldots, \xi_x^l) = [(u, (H^*(\mathrm{d}\,\hat{\alpha} + \hat{\gamma} \bullet \hat{\alpha}))(\xi_u^o, \ldots, \xi_u^l))]$$

where $p(u) = x$ and $Tp(\xi_u^i) = \xi_x^i$ $i = 0, \ldots l, \hat{\alpha}$ and $\hat{\gamma}$ are the equivariant $\mathbf{g}$-valued forms on P representing α and γ respectively. Moreover the dot is the following wedge product:

$$\bullet : A^k(P, \mathbf{g}) \times A^l(P, \mathbf{g}) \to A^{k+l}(P, \mathbf{g})$$

$$(a \otimes h) \bullet (b \otimes k) = a \wedge b \otimes [h, k]$$

for any $a \in A^k(P), b \in A^l(P)$ $h, k \in g$ and $A_l(P,g)$ is identified with $A_l(P) \otimes g$.

One can see that for $s \in \mathcal{G}$

$$D^{s\gamma} \mathbf{Ad}_s \alpha = \mathbf{Ad}_s D^\gamma \alpha$$

where we denote by $\mathbf{Ad}_s$ the extensions of the map $\mathbf{Ad}_s$ defined in Section IV to the entire $A(M, P[g])$.

For every $k \in \mathbb{N}$, the wanted $\mathcal{G}$-invariant metric on C is defined by

$$(\alpha|\beta)^k_\gamma = \sum_{l=0}^{k} \int_M ((D^\gamma)^l \alpha(x)|(D^\gamma)^l \beta(x))_x \mathrm{d}\, m$$

for $\alpha, \beta \in \mathcal{A}$, $\gamma \in C$. Here we make use of a Riemannian metric on the bundle $(\otimes^l T^* M) \otimes P[g] = L(\otimes^l TM, P[g])$ defined by a metric on M and the $\mathbf{Ad}$ -invariant scalar product on g as in formula (4.2).

The metrics $(\,|)^k_\gamma$'s are smooth tame since the family of linear maps D^γ's are smooth tame maps on $C \times A^l(M, P[g])$ to $A^{l+1}(M, P[g])$ (see [13] II 3.3). For a fixed connection γ the scalar products $\{(\,|)^k_\gamma\}(k \in \mathbb{N})$ define on $\mathcal{A}$ an S_γ-invariant grading equivalent to the Sobolev grading [9].

Now we are ready to approch the slice theorem. Let O_γ be the orbit through γ. We consider the normal bundle $N(O_\gamma) = N_\gamma$ consisting of all vectors of $TC_{\lceil O_\gamma}$ orthogonal to TO_γ with respect to the weak Riemannian metric (4.2). Using the procedure of Lemma 4.4 one can easily recognize that N_γ is a $\mathcal{G}$-invariant smooth tame subbundle of $TC_{\lceil O_\gamma}$.

The map

$$\Sigma : TC \to C, \Sigma(\gamma, \alpha) = \gamma + \alpha \quad \gamma \in C, \alpha \in \mathcal{A}$$

is a $\mathcal{G}$-invariant tame local addition on C.

Tameness properties of Σ and N_γ allows us to apply Nash-Moser theorem and to prove that, if restricted to N_γ, Σ is a local diffeomorphism at every point of the zero section.

As O_γ is a closed submanifold of a manifold admitting partitions of unity and is paracompact and normal (see [18] 4.11), we can rephrase Th. 9 in Sec. 5 of [22] and prove that the map Σ and the vector bundle N_γ realize a tubular neighborhood of O_γ, i.e. the map Σ is a diffeomorphism of an neighborhood in N_γ of the zero section onto an open subset of C containing the orbit O_γ.

Now, by the properties of the above defined metrics on C one can easily see that the family of open subsets of N_γ

$$N^k_{\gamma,\epsilon} = \{\xi_{\gamma'} \in N_\gamma, (\xi_{\gamma'}|\xi_{\gamma'})^k_{\gamma'} < \epsilon\} \ \epsilon > 0, k \in \mathbb{N}$$

is a basis of $\mathcal{G}$-invariant neighborhoods of the zero section.

We can restrict once more the map Σ to a suitable tube $N^k_{\gamma,\epsilon}$ obtaining an equivariant diffeomorphism from $N^k_{\gamma,\epsilon}$ to an open $\mathcal{G}$-invariant neighborhood of $\mathcal{O}_\gamma$.

Thus the composition $\tau = \Sigma \circ \rho^{\epsilon,k}$ where

$$\rho^{\epsilon,k} : N_\gamma \to N^k_{\gamma,\epsilon} \quad \rho^{\epsilon,k}(\xi_{\gamma'}) = \epsilon((\xi_{\gamma'}|\epsilon_{\gamma'})^k_{\gamma'} + 1)^{-\frac{1}{2}}\epsilon_{\gamma'}$$

is a $\mathcal{G}$-invariant tame diffeomorphism from N_γ onto an open $\mathcal{G}$-invariant neighborhood of the orbit.

The tame splitting submanifold

$$\mathcal{S}_\gamma = \tau(T_\gamma\mathcal{C} \cap N_\gamma)$$

is the slice at γ. Conditions i) and ii) of Definition 5.1 are easily proved by equivariance of τ and by the $\mathcal{S}_\gamma$-invariance of the splitting in Theorem 4.3. We prove the third condition. Let $\chi : U \subset \mathcal{G}/\mathcal{S}_\gamma \to \mathcal{G}$ be a local section as in Section IV. Then $i_\gamma(U)$ is an open neighborhood of γ in $\mathcal{O}_\gamma$. We prove that the map

$$\Phi : U \times \mathcal{S}_\gamma \to \tau(N_{\gamma \lceil i_\gamma(U)}) \quad \Phi(u, \gamma') = \chi(u)\gamma'$$

is a tame diffeomorphism. If $\gamma' \in \mathcal{S}_\gamma$, then $\gamma' = \tau(\gamma, \alpha)$ for $\alpha \in (\mathrm{Im}\,\nabla_\gamma)^\perp$. Therefore

$$\Phi(u, \gamma') = \chi(u)\tau(\gamma, \alpha) = \tau(\chi(u)\gamma, \mathbf{Ad}_{\chi(u)}\,\alpha) = \tau(i_\gamma(u), \mathbf{Ad}_{\chi(u)}\,\alpha)$$

for every $u \in U, \gamma' \in \mathcal{S}_\gamma$. Then Φ is obtained by composition of tame diffeomorphisms.

We have the following theorem.

THEOREM 5.1. *At every $\gamma \in \mathcal{C}$, there exists a slice $\mathcal{S}_\gamma$ for the action of $\mathcal{G}$ on $\mathcal{C}$ with the properties:*

i) $\mathcal{S}_\gamma$ *is a tame splitting submanifold of* $\mathcal{C}$*;*

ii) Φ *is a tame diffeomorphism.* ∎

VI. CONSEQUENCES OF THE SLICE THEOREM

Here we only draw some of the most important consequences of the slice theorem which concern the structure of the orbit space. Most of these consequences stem essentially from the general meaning of the theorem rather than from the particular context.

As a first consequence of the slice theorem we are able to define on $\mathcal{C}$ a $\mathcal{G}$-invariant smooth tame stratification. We recall that a smooth stratification of a Hausdorff topological space X is a countable partition $\{X_i\}_{i\in I}$, where X_i are smooth manifolds in the relativized topology and the frontier property

$$(6.1) \qquad X_i \cap \overline{X_j} \neq \emptyset \quad \text{implies} \quad X_i \subset \overline{X_j}$$

holds.

The manifold X_i is called stratum ([8], [10], [11], [13]).

To construct in $\mathcal{C}$ the wanted stratification, we denote by I the set of orbits types, namely the set of conjugacy classes (S) of closed subgroups S of $\mathcal{G}$ which are stability subgroups; by $\mathcal{C}(S)$ we denote the $\mathcal{G}$-invariant subset of connections having a stability subgroup belonging to (S). Then $\{\mathcal{C}(S)\}$ $((S) \in I)$ is obviously a $\mathcal{G}$-invariant partition of $\mathcal{C}$.

The cardinality of I does not exceed the sum of cardinalities of isomorphism classes of reductions of $P(M, G)$, where the sum is labelled by conjugation classes of subgroups of the form $\mathcal{C}^2(\mathcal{H}_{u_o})$ (double centralizer of an holonomy group $\mathcal{H}_{u_o}$), see Th. 4.2.1. [13].

Countability of such isomorphism classes follows by the general result that isomorphism classes of principal bundles with a given structure group and a given base manifold are countable and from the countability of the conjugation classes of the subgroup under consideration. Under our assumption on the structure group G we get indeed $G = K \times \mathbb{R}^n$ where K is a compact Lie group. If G_o is a subgroup of G, its centralizer $C(G_o)$ has the form $K_o \times \mathbb{R}^n$ for some closed subgroup K_o of K. Moreover G-conjugation classes of centralizers correspond injectively with K-conjugation classes of closed subgroups of K. Countability of this last set was proved in [23].

As a first consequence of the slice theorem, we prove that every stratum is a tame splitting submanifold of $\mathcal{C}$. Let S be the stability subgroup of a connection γ. We recall that $\tau(N_\gamma)$ is an open neighborhood of γ in $\mathcal{C}$, so that we have just to prove that $\tau(N_\gamma \cap \mathcal{C}(S))$ is a tame splitting submanifold of $\tau(N_\gamma)$.

To see this fact remark that $S_{\gamma'} \subset S$, for every $\gamma' \in \mathcal{S}_\gamma$. This implies that $\mathcal{S}_\gamma \cap \mathcal{C}(S)$ is the subset $\widetilde{\mathcal{S}_\gamma} = \{\gamma' \in \mathcal{S}_\gamma | S_{\gamma'} = S\}$, so that $\tau(N_\gamma) \cap \mathcal{C}(S)$ is precisely $\tau(\tilde{N}_\gamma)$, where $\tilde{N}_\gamma = \{(\gamma', \xi') \in N_\gamma \mid \tau(\gamma', \xi') \in \tilde{\mathcal{S}}_{\gamma'}\}$. This latter is a tame subbundle of N_γ with fiber the tame subspace $\tilde{\mathcal{F}}_\gamma$ of the fiber at γ of the normal bundle N_γ, consisting of the vectors ξ whose stability subgroup for the action $\mathbf{Ad}$ is precisely S.

In fact $\tilde{N}_\gamma$ is a trivial subbundle with trivializing map

$$\tilde{\psi} : \tilde{N}_\gamma \to \mathcal{O}_\gamma \times \tilde{\mathcal{F}}_\gamma, \tilde{\psi}(\gamma', \xi') = (\gamma', \mathbf{Ad}_{s^{-1}} \xi')$$

where $s \in \mathcal{G}$ is such that $s\gamma = \gamma'$. In particular $\tilde{N}_\gamma$ is smooth tame submanifold in N_γ and this implies that $\mathcal{C}(S)$ is a tame Fréchet manifold.

To prove the frontier property (6.1) the relevant point is that the closure $\bar{C}(S)$ of $C(S)$ agrees with $\bigcup\{C(S')|(S) < (S')\}$, where $(S) < (S')$ if there is in (S) a subgroup of S' [14]. Then condition (6.1) follows immediately since

$$C(S') \cap \bar{C}(S) \neq \emptyset \Rightarrow (S) < (S')$$

so that $C(S') \subset \bar{C}(S)$.

It is whortwile to remark that $C(S)$ is open in $\bar{C}(S)$, since for $\gamma \in C(S)$ the open subset $\tau(N_\gamma) \cap \bar{C}(S)$ is contained in $C(S)$. Therefore $C(S)$ is a residual set in $\bar{C}(S)$. In particular if Z is the centrum of G, the stratum of irreductible connections $C(Z)$ is a residual set in $C = \bar{C}(Z)$ (in physical literature this is referred as a generic set).

We can therefore conclude that $\{C(S)\}((S) \in I)$ is a $\mathcal{G}$-invariant stratification of C, where the strata are tame Fréchet manifolds.

Coming to the orbit space $\mathcal{R} = C/\mathcal{G}$ itself, one easily proves by topological arguments and by the slice theorem that $\mathcal{R}$ is a connected regular second countable space, hence metrizable. We can prove that the set $\mathcal{R}(S) = C(S)/\mathcal{G}$ is a tame Fréchet manifold, for every $(S) \in I$.

Actually, for $\gamma \in C(S)$, consider the $\mathcal{G}$-invariant tubular neighborhood $\tau(\tilde{N}_\gamma)$. A chart on the open neighborhood $\tau(\tilde{N}_\gamma)/\mathcal{G}$ of $\mathcal{O}_\gamma$ in $\mathcal{R}(S)$ is given by the quotient of the $\mathcal{G}$-invariant map $\tilde{\psi}_\gamma$, where

$$\tilde{\psi}_\gamma : \tau(\tilde{N}_\gamma) \to \tilde{\mathcal{F}}_\gamma, \tilde{\psi}_\gamma := pr_2(\tilde{\psi} \circ \tau^{-1})$$

and $\tilde{\psi}$ is the trivializing map for $\tilde{N}_\gamma$. One easily checks that these quotient maps (as γ ranges in $C(S)$) give a system of charts for the structure of tame manifold on the quotient space $\mathcal{R}(S)$. $\mathcal{R}(S)$ is modelled on a nuclear Fréchet space and therefore is paracompact. The quotient map $\pi_{(s)} : C(S) \to \mathcal{R}(S)$ is a smooth tame surjection and $(C(S), \pi_{(S)}, \mathcal{R}(S); \mathcal{G}/S)$ is a tame fiber bundle with typical fiber $\mathcal{G}/S$.

The $\mathcal{G}$-invariant stratification of C gives, by simple topological arguments, a stratification $\{\mathcal{R}(S)\}((S) \in I)$ of the quotient space $\mathcal{R}$. The stratum $\mathcal{R}(S)$ is open dense in $\bigcup\{\mathcal{R}(S')|(S) < (S')\}$. In particular, $\mathcal{R}(Z) = C(Z)/\mathcal{G}$ is a generic subset of $\mathcal{R}$.

It has been stressed by [3], [4] and [5] that a natural weak Riemannian metric is defined on the stratum $\mathcal{R}(Z)$ and that there is a deep link between the usual Faddeev-Popov determinant and the determinant of this metric. More generally, a weak metric can be introduced on each stratum in the stratification of Sobolev connections [15]. In an analogous way we can introduce a weak Riemannian metric on each stratum $\mathcal{R}(S)$. This metric will be defined by means of the weak $\mathcal{G}$-invariant metric (4.2) and the help of a $\mathcal{G}$-invariant connection 1-form on the bundle $(C(S), \pi_{(S)}, \mathcal{R}(S); \mathcal{G}/S)$. In a general finite dimensional fiber bundle one can develop a generalized connection theory, as summarized in [24]. According to this theory we define, even in a infinite dimensional fiber bundle $(E, \pi, X; F)$ a connection 1-form as a vector bundle morphism $\phi : TE \to TE$ with the properties

i) $\phi = \phi^2$

ii) $\operatorname{Im}\phi$ coincides with the vertical bundle $\operatorname{Ver} E$.

The horizontal lift defined by ϕ is the smooth map

$$C : TX \times_X E \to TE, \quad C(v_x, u) = (1 - \phi)(w_u)$$

where $T\pi(w_u) = v_x$; the map C is well defined since $T\pi(w'_u) = v_x$ implies $(1 - \phi)(w_u) = (1-\phi)(w'_u)$. As in the ordinary case the corrispondence between connection 1-forms and horizontal lifts is a bijection. However, horizontal lifts do not guarantee the parallel trasport of curves. This difficulty arises from the lack of inverse function theorem. Moreover we recall that for infinite dimensional vector bundles the kernel of a bundle morphism can fail to be a subbundle [25], so that the horizontal bundle could not exist.

To define a connection 1-form $\phi^{(s)}$ on the stratum $\mathcal{C}(S)$ we first introduce a vector bundle morphism $\phi : T\mathcal{C} = \mathcal{C} \times \mathcal{A} \to T\mathcal{C} = \mathcal{C} \times \mathcal{A}$ defined by $\phi(\gamma, \alpha) = (\gamma, P_\gamma \alpha)$, where for $\gamma \in \mathcal{C}, P_\gamma$ denotes the orthogonal projection on the tame splitting subspace $\operatorname{Im}\nabla_\gamma$. The restriction of ϕ to $T\mathcal{C}(S)$ induces a vector bundle morphism $\phi^{(s)} : T\mathcal{C}(S) \to T\mathcal{C}(S)$, since for $\gamma \in \mathcal{C}(S), \operatorname{Im}\nabla_\gamma$ is a subspace of $T_\gamma\mathcal{C}(S)$. Actually, $\operatorname{Im}\nabla_\gamma$ is tangent to the orbit and $\mathcal{C}(S)$ is a $\mathcal{G}$-invariant submanifold.

To prove that $\phi^{(s)}$ is a connection 1-form on the fiber bundle $(\mathcal{C}(S), \pi_{(s)}, \mathcal{R}(S); \mathcal{G}/S)$ we have just to prove that $\phi^{(s)}$ is smooth, since properties i) and ii) are trivially verified.

For every $\gamma \in \mathcal{C}$, the projection P_γ can be decomposed as $P_\gamma = \nabla_\gamma G_\gamma \nabla^\dagger_\gamma$, where $\nabla^\dagger_\gamma$ is the adjoint operator of ∇_γ with respect to the weak scalar products (4.1) and (4.2), and C_γ is the Green operator of the elliptic operator $\nabla^\dagger_\gamma\nabla_\gamma : L(\mathcal{G}) \to L(\mathcal{G})$ [15]. Hence smoothness of ϕ and $\phi^{(s)}$ follows by smoothness of the map $\mathcal{C} \times \mathcal{A} \ni (\gamma, \alpha) \to P_\gamma\alpha = \nabla_\gamma G_\gamma \nabla^\dagger_\gamma \in \mathcal{A}$.

One easily recognizes that the maps $(\gamma, \lambda) \mapsto \nabla_\gamma\lambda$ and $(\gamma, \alpha) \mapsto \nabla^\dagger_\gamma$ are smooth tame. Actually, by [26] and II.3 of [12] they are smooth tame families of differential operators. Smoothness and tameness of the map $(\gamma, \lambda) \to G_\gamma(\lambda)$ follows by Theorem II, 3.3.3 of [12]. By composition we obtain the smoothness and tameness of $\phi^{(s)}$.

By formulae (4.3) and (4.5) we see that $\phi^{(s)}$ is a $\mathcal{G}$-invariant connection 1-form.

Finally, using the horizontal lift C defined by $\phi^{(s)}$ we can introduce on $\mathcal{R}(S)$ the tame weak metric

$$(\xi_{o_\gamma}|\xi'_{o_\gamma})^{(S)}_{o_\gamma} = (C(\xi_{o_\gamma}, \gamma), C(\xi'_{o_\gamma}, \gamma))_\gamma.$$

The invariance of the metric (4.2) and of the connection 1-form $\phi^{(S)}$ guarantee that this metric is well defined.

Thus, $\mathcal{R}(S)$ admits a very natural tame weak Riemannian metric, which is a natural extension of the metric discussed by Singer [5] and by Babelon and Viallet [3] for generic connections. To compose with horizontal lift amounts to remove vertical components of fields on true configuration space and the unphysical degrees of freedom corresponding to gauge transformations. It is therefore natural to assume that this weak Riemannian metric on $\mathcal{R}(S)$ provides the kinetic part of the Lagrangian on the true configuration space [3].

To conclude, $\mathcal{R}$ *is a connected metrizable second countable space and the family* $\{\mathcal{R}(S)\}((S) \in I)$ *is a stratification for* $\mathcal{R}$, *where the strata are tame Fréchet manifolds admitting a tame weak Riemannian metric.*

APPENDIX I

THEOREM A.I.1. *Let* G *be a separable metrizable and complete topological group acting continuosly and transitively on a first countable non meagre Hausdorff topological space* $\mathcal{O}$. *For every* $o \in \mathcal{O}$ *the map* $A_o : G \to \mathcal{O}, A_o(g) = go$ *is open.*

The proof follows by the two next lemmas.

LEMMA A.I.2. *Let* A *be a transitive continuous action of a separable topological group* G *on a non meagre topological space* $\mathcal{O}$. *Then for every neighborhood* θ *of the unit* e *of* G *and* $o \in \mathcal{O}$, *the set* $\overline{\theta o}$ *is a neighborhood of* o.

Proof. By separability of G, there exists a countable subset $\{g_n\}$ of G such that for every open neighborhood U of the unit, the family $\{g_n U\}$ is a covering of G. By transitivity of the action, the countable family $\{g_n U o\}$ covers the non meagre set $\mathcal{O}$ by sets, all homeomorphic to Uo. This implies that Uo is a non rare set. Therefore, there exist $g \in U$ such that $go \in \mathrm{int}(\overline{Uo})$ so that $o \in \mathrm{int}(\overline{g^{-1}Uo}) \subset \mathrm{int}(\overline{U^{-1}Uo})$.

Finally, for every neighborhood θ of the unit of G, there exists a neighborhood U of the unit such that $U^{-1}U \subset \theta$, so that $o \in \mathrm{int}(\overline{\theta o})$. ∎

LEMMA A.I.3. *Let* G *be a metrizable separable and complete topological group and* $\mathcal{O}$ *be a first countable Hausdorff topological space. If for every neighborhoods* θ *of* e *and some* $o \in \mathcal{O}$ *the set* $\overline{\theta o}$ *is a neighborhood of* o, *then the map*

$$A_o : G \to \mathcal{O}, \quad A_o(g) = go$$

is open.

Proof. Let $\{U_n\}$ be a countable basis of open neighborhoods of e such that $U_{n+1}U_{n+1} \subset U_n$. By assumption, $W_n := \overline{A_o(U_n)}$ is a neighborhood of o. We can assume that $V_n \subset W_n$, where $\{V_n\}$ is a basis of open neighborhoods of o. To prove our lemma will be enough to prove that $W_1 \subset A_o(U_o)$.

By assumption, the family $\{gV_{n+1}, g \in U_n\}$ covers W_n. Therefore, for every $\omega_1 \in W_1$ there exists $\omega_2 \in V_2 \subset W_2$ and $g_1 \in U_1$ such $\omega_1 = g_1\omega_2$. By induction, we can construct a sequence $\{\omega_n\}$ such that

$$\omega_1 = g_1 g_2 \cdots g_{n-1}\omega_n, \quad \omega_n \in V_n \subset W_n,\ g_i \in U_i.$$

The sequence $\{h_n\} = \{g_1 \dots g_n\}$ is a Cauchy sequence converging to some $g_0 \in U_0$ and ω_n converges to o. We conclude that $\omega_1 = g_0 o = A_o(g_0)$ with $g_0 \in U_0$. ■

APPENDIX II

The examples we give here concern a trivial principal bundle $P = S^1 \times G$ with G a matrix group. The group $\mathcal{G}$ is $C^\infty(S^1, G)$, its Lie algebra is $C^\infty(S^1, \mathbf{g})$ and C can be identified with $C^\infty(S^1, \mathbf{g})$ by $\gamma(t) \equiv L(t)\mathrm{d}t, L \in C^\infty(S^1, \mathbf{g})$. This case was investigated in [27] and [28] identifying the gauge transformation action with the coordinate change in an auxiliary equation $\dot{x} = L(t)x$. The authors investigated the monodromy map $T : C^\infty(S^1, \mathbf{g}) \to G$ which associates to every $L \in C^\infty(S^1, \mathbf{g})$ its monodromy operator and proved that this map establishes a injective correspondence between the gauge transformation orbits and the conjugacy classes of monodromy operators. We improve their result by the following theorem.

THEOREM A.II.1. *The map T is continuous.*

Proof. We recall that the monodromy operator $T(L)$ is obtained by evaluation at $t = 2\pi$ of the fundamental matrix $\Phi(t)$ of solutions for the auxiliary equation. On the other hand, $\Phi(t)$ depends continuously on L. In fact, the elements $L_{ij}(t)$ of the matrix $L(t)$ can be interpreted as coefficients of the auxiliary equation belonging to a complete metric space (they belong indeed to $C^\infty(S^1, \mathbf{R})$ or $C^\infty(S^1, \mathbb{C})$) and a well known theorem on differential equations assures that solutions depends continuously by such coefficients. Hence the map T is a composition of continuous maps. ■

Continuity of T assures that gauge transformations orbits are closed or locally closed whenever the orbits of the action of G on itself by inner automorphisms are closed or locally closed. In this way one can for instance prove that the gauge transformation orbits are closed if G is an Euclidean group. To give an example of a not closed gauge transformation orbit we consider briefly the case $G = GL(2, \mathbf{R})$.

Consider

$$\gamma \in \mathcal{C} \equiv C^\infty(S^1, M(2, \mathbf{R})), \quad \gamma(t) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and the sequence

$$s_n \in \mathcal{G} \equiv C^\infty(S^1, GL(2, \mathbf{R})), \quad s_n(t) = \begin{pmatrix} 1 & \frac{1}{n} \\ -1 & \frac{1}{n} \end{pmatrix}.$$

Then $s_n^{-1}\gamma = \gamma_n$ where

$$\gamma_n(t) = s_n(t)\gamma(t)s_n^{-1}(t) + \dot{s}_n(t)s_n^{-1} = \frac{1}{2}\begin{pmatrix} \frac{1}{n} & -\frac{1}{n} \\ \frac{1}{n} & -\frac{1}{n} \end{pmatrix}.$$

Obviously

$$\gamma_n \to \gamma_o \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

but γ_o does not belong to the orbit of γ. However the orbits are locally closed. One can easily examine directly the conjugacy classes of $GL(2, \mathbf{R})$ and prove that they are locally closed. The elements of $GL(2, \mathbf{R})$ having distinct eigenvalues generate closed orbits. In the case of a degenerate eigenvalue λ one has the singular point $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ or the orbit generated by the element $\begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$. This orbit is not closed but locally closed. In fact its closure is obtained adding only the singular point.

REFERENCES

[1] R. CIRELLI and A. MANIÀ: *The group of gauge transformations as a Schwartz-Lie group*, J. Math. Phys. **26** (12) (1985) 30-36.

[2] M.C. ABBATI, R. CIRELLI, A. MANIÀ and P.W. MICHOR: *Smoothness of the action of gauge transformation group on connections*, J. Math. Phys. **27** (10), (1986), 2469-2474.

[3] O. BABELON and C.M. VIALLET: *The Riemannian Geometry of the Configuration Space of Gauge Theories*. Commun. Math. Phys. **81** (1981) 515-525.

[4] I.M. SINGER: *Some remarks on the Gribov ambiguity*. Commun. Math. Phys. **60** (1978), 7-12.

[5] I.M. SINGER: *The Geometry of the Orbit space for Non-Abelian Gauge Theories*. Physica Scripta, Vol. 24, (1981), 817-820.

[6] O. BABELON and C.M. VIALLET: *The Geometric Interpretation of the Faddev-Popov Determinant*. Phys. Letters **85** B (2), (1979).

[7] M. ASOREY and F. FALCETO: *Geometry Regularization of Gauge Theories*. preprint HUTP-88 (1988).

[8] A. FISCHER: *The theory of superspace*, in «Relativity», M. Carmeli, S. Fickler and L. Witten eds. (Plenum, 1967).

[9] D.G. EBIN: *The manifold of Riemannian metrics*, Proc. Symp. Pure Math. A.M.S. XV (1970).

[10] J.P. BOURGUIGNON: *Une stratification de l'éspace des structures riemanniennes*. Comp. Math. **30** (1975), 1-41.

[11] J. ISENBERG and J.E. MARSDEN: *A slice theorem for the space of solutions of Einstein's equations*. Physics Reports **89** N.2 (1982), 179-222.

[12] R.S. HAMILTON: *The inverse function theorem of Nash and Moser*. Bull. Amer. Math. Soc. 7, (1982) 65-222.

[13] W. KONDRACKI and J. ROGULSKI: *On the stratification of the orbit space for the action of automorphisms on connections*. Institute of Mathematics, Polish Academy of Sciences, Warsawa, Preprint IMPAN 281 (1983).

[14] W. KONDRACKI and P. SADOWSKI: *Geometric structure of the orbit space of gauge connections*. Institute of Mathematics, Polish Academy of Sciences, Warsawa, Preprint IMPAN 36/168 (1984).

[15] V. BERZI and M. RENI: *Weak Riemannian Structures on Gauge-Group Orbits*. Int. J. Theor. Phys. **26** n.2 (1987), 151-174.

[16] M.F. ATIYAH and R. BOTT: *Yang-Mills equations over Riemann surfaces*. Phil. Trans. R. Soc. London, **A 308** (1982), 523-615.

[17] J.M. ARMS, J.E. MARDSEN and V. MONCRIEF: *Simmetry and Bifurcations of Momentum Mappings*. Commun. Math. Phys. **78** (1981), 455-478.

[18] P.W. MICHOR: *Manifolds of differentiable mappings*. Shiva Matematics Series [**3**] (Shiva Publ. Lmd. Orpington, Kent, 1980).

[19] W. GREUB, S. HALPERIN and R. VANSTONE: *Connections, Curvature, and Cohomology*. **I, II** (Acad. Press N.Y., 1972).

[20] M. CANTOR: *Elliptic operators and the decomposition of tensor fields*, Bull. Amer. Math. Soc. **5** n.3 (1981) 235-262.

[21] R.S. PALAIS: *The classification of G-spaces*. Ann. Am. Math. Soc. **36** (1960).

[22] S. LANG: *Differentiable Manifolds*. (Addison Wesley, Reading 1972).

[23] W. KONDRACKI and J. ROGULSKI: *On conjugacy classes of subgroups*. Polish Academy of Sciences, Warsawa, Preprint 281 IMPAN (1983).

[24] P. MICHOR: *Gauge theory for diffeomorphism groups*. Proceeding of the conference on «Differential geometric methods in theoretical physics» Como 1987 (D. Reidel, Dordrecht 1988).

[25] N. BOURBAKI: *Éléments de mathématique*. Fascicule XXXIII, Variétés différentiel les et analitiques. Fascicule de résultats.

[26] R.S. PALAIS: *Seminar on the Atiyah-Singer Index Theorem*. Annals of Math. Studies **57** (Princeton University Press, Princeton 1965).

[27] A.G. REIMAN and M.A. SEMENOV-TJAN-SANSKII: *Current algebras and non linear partial differential equations*. Sov. Math. Dokl. **2** (1980), 630-634.

[28] M.A. SEMENOV-TJAN-SANSKII: *Group theoretical aspects of completely integrable systems*. in L.N.M. 970 (Springer N.Y. 1983).

Manuscript received: February 13, 1989